On $\psi\alpha g$-Separation axioms in Topological Spaces

V. Kokilavani, P.R. Kavitha

1. Assistant professor, Dept. of Mathematics, Kongunadu Arts and Science College, Coimbatore, Tamilnadu, India
2. Assistant professor, Dept. of Mathematics, Kongunadu Arts and Science College, Coimbatore, Tamilnadu, India

Abstract: In the present paper, we introduce, study and investigate the following separation axioms: $\psi\alpha g$ - $T_i$ spaces (for $i=0,1,2$). Moreover, some of special results and properties, which belong to them, are studied.

Keywords: $\psi\alpha g$-closed; $\psi\alpha g$-$T_0$; $\psi\alpha g$-$T_1$; $\psi\alpha g$-$T_2$.

1. INTRODUCTION

Recently V. Kokilavani and P.R. Kavitha [12] defined the concept of $\psi\alpha g$-closed sets and studied some of their properties.

The aim of this paper is to introduce a new type of function is called $\psi\alpha g$-open and quasi $\psi\alpha g$-closed functions, quasi $\psi\alpha g$-open and quasi $\psi\alpha g$-closed functions. Also, we obtain its characterizations and its basic properties and we study a new type of weak separation axioms, namely $\psi\alpha g$-$T_0$, $\psi\alpha g$-$T_1$, $\psi\alpha g$-$T_2$ and separation properties obtained by utilizing $\psi\alpha g$-closed sets.

2. PRELIMIERIES

Definition: 2.1 Let $(X, \tau)$ be a topological space. A subset $A$ of the space $X$ is said to be

(i) semi open set [2] if $A \subseteq cl(int(A))$.

(ii) $\alpha$-open set [4] if $A \subseteq int(cl(int(A)))$.

The complements of the above mentioned sets are called their respective closed sets. The $\psi$-closure of a subset $A$ of a space $(X, \tau)$ is the intersection of all $\psi$-closed sets that contain $A$ and is denoted by $\psi cl(A)$. The $\psi$-interior of a subset $A$ of a space $(X, \tau)$ is the union of all $\psi$-open sets contained in $A$ and is denoted by $\psi int(A)$.

Definition: 2.2 A subset $A$ of a space $X$ is $\psi\alpha g$-closed if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha g$-open in $X$. The family of all $\psi\alpha g$-closed subsets of the space $X$ is denoted by $\psi\alpha g C(X)$.

Definition: 2.3 The intersection of all $\psi\alpha g$-closed sets containing a set $A$ is called $\psi\alpha g$-closure of $A$ and is denoted by $\psi\alpha g - cl(A)$. A set $A$ is $\psi\alpha g$-closed set if and only if $\psi\alpha g - cl(A) = A$.

Definition: 2.4 The union of all $\psi\alpha g$-open sets contained in $A$ is called $\psi\alpha g$-interior of $A$ is denoted by $\psi\alpha g - int(A)$. A set $A$ is $\psi\alpha g$-open sets if and only if $\psi\alpha g - int(A) = A$.

A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called

(i) $g$-continuous[2] if $f^{-1}(V)$ is a $g$-closed in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$.

(ii) $\alpha$-continuous[15] if $f^{-1}(V)$ is a $\alpha$-closed in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$.

(iii) $\alpha g$-continuous[6] if $f^{-1}(V)$ is a $\alpha g$-closed in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$.

(iv) $\psi\alpha g$-continuous[6] if $f^{-1}(V)$ is a $\psi\alpha g$-closed in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$.

(v) $\alpha$- irresolute[10] if $f^{-1}(V)$ is a $\alpha$-open set in $(X, \tau)$ for every $\alpha$-open set $V$ of $(Y, \sigma)$.

(vi) $\alpha$-quotient map[10] if $f$ is $\alpha$-continuous and $f^{-1}(V)$ is open set in $(X, \tau)$ implies $V$ is an $\alpha$-open set in $(Y, \sigma)$.

(vii) Weakly continuous[9] if for each point $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing $x$ such that $f(U) \subseteq cl(V)$.

3. Quasi $\psi\alpha g$-open functions

We introduce a new definition as follows:

Definition: 3.1 A function $f:X \rightarrow Y$ is said to be quasi $\psi\alpha g$-open if the image of every $\psi\alpha g$-open set in $X$ is open in $Y$.

It is evident that, the concepts quasi $\psi\alpha g$-openness and $\psi\alpha g$-continuity coincide if the function is a bijection.

Theorem: 3.2 A function $f:X \rightarrow Y$ is quasi $\psi\alpha g$-open if and only if for every subset $U$ of $X$, $f(\psi\alpha g - int(U)) \subseteq int(f(U))$.

Proof: Let $f$ be a quasi $\psi\alpha g$-function. Now, we have $int(U) \subseteq U$ and $\psi\alpha g$-$int(U)$ is a $\psi\alpha g$-open set. Hence, we obtain that $f(\psi\alpha g - int(U)) \subseteq f(U)$. As $f(\psi\alpha g - int(U))$ is open, $f(\psi\alpha g - int(U)) \subseteq int(f(U))$. Conversely, assume that $U$ is a $\psi\alpha g$-open set in $X$. Then,
Theorem: 3.3 A function \( f: X \to Y \) is quasi \( \psi ag \)-open, then \( \psi ag - Int(f^{-1}(G)) \subset f^{-1}(Int(G)) \) for every subset \( G \) of \( Y \).

**Proof:** Let \( G \) be any arbitrary subset of \( Y \). Then, \( \psi ag - Int(f^{-1}(G)) \) is a \( \psi ag \)-open set in \( X \) and \( f \)

\( \psi ag \)-open, then \( f(\psi ag - Int(f^{-1}(G))) \subset Int(f^{-1}(Int(G))) \subset Int(G) \). Thus, \( \psi ag - Int(f^{-1}(G)) \subset f^{-1}(Int(G)) \).

Recall that a subset \( S \) is called a \( \psi ag \)-neighbourhood of a point \( x \) of \( X \) if there exists a \( \psi ag \)-open set \( U \) such that \( x \in U \subset S \).

Theorem: 3.4 For a function \( f: X \to Y \), the following are equivalent:

(i) \( f \) is quasi \( \psi ag \)-open;

(ii) For each subset \( U \) of \( X \), \( f(\psi ag - Int(U)) \subset Int(f(U)) \);

(iii) For each \( x \in X \) and each \( \psi ag \)-neighbourhood \( U \) of \( x \) in \( X \), there exists a neighbourhood \( f(U) \) of \( f(x) \) in \( Y \) such that \( V \subset f(U) \).

**Proof:** (1) \( \Rightarrow (2) \): It follows from Theorem 3.2.

(2) \( \Rightarrow (3) \): Let \( x \in X \) and \( U \) be an arbitrary \( \psi ag \)-neighbourhood \( U \) of \( x \) in \( X \). Then there exists a \( \psi ag \)-open set \( V \subset X \) such that \( x \in V \subset U \). Then by (ii), we have \( f(V) = f(\psi ag - Int(V)) \subset Int(f(U)) \) and hence \( f(V) = Int(f(V)) \).

Therefore, it follows that \( f(V) \) is open in \( Y \) such that \( f(x) \in f(V) \subset f(U) \).

(3) \( \Rightarrow (1) \): Let \( U \) be an arbitrary \( \psi ag \)-open set in \( X \). Then for each \( y \in f(U) \), by (iii) there exists a neighbourhood \( V_y \) of \( y \) in \( Y \) such that \( V_y \subset f(U) \).

As \( V_y \) is a neighbourhood of \( y \), there exists an open set \( W_y \) in \( Y \) such that \( y \in W_y \subset V_y \).

Thus \( f(U) = \bigcup \{W_y; y \in f(U)\} \) which is an open set in \( Y \). This implies that \( f \) is quasi \( \psi ag \)-open function.

Theorem: 3.5 A function \( f: X \to Y \) is quasi \( \psi ag \)-open if and only if for any subset \( B \) of \( Y \) and for any \( \psi ag \)-closed set \( F \) of \( X \) containing \( f^{-1}(B) \), there exists a closed set \( G \) of \( Y \) containing \( B \) such that \( f^{-1}(G) \subset F \).

**Proof:** Suppose \( f \) is \( \psi ag \)-open. Let \( B \subset Y \) and \( F \) be a \( \psi ag \)-closed set of \( X \) containing \( f^{-1}(B) \). Now put \( G = Y - f(X - F) \). It is clear that \( f^{-1}(G) \subset F \) implies \( B \subset G \). Since \( f \) is \( \psi ag \)-open, we obtain \( G \) as a closed set of \( Y \). Moreover, we have \( f^{-1}(G) \subset F \).

Conversely, let \( U \) be a \( \psi ag \)-open set of \( X \) and put \( B = Y \setminus f(U) \). Then \( X \setminus U \) is a \( \psi ag \)-closed set in \( X \) containing \( f^{-1}(B) \). By hypothesis, there exists a closed set \( F \) of \( Y \) such that \( B \subset F \) and \( f^{-1}(F) \subset X \setminus U \). Hence, we obtain \( f(U) \subset Y \setminus F \). On the other hand, it follows that \( B \subset F \), \( Y \setminus F \subset Y \setminus B = f(U) \). Thus, we obtain \( f(U) \subset Y \setminus F \) which is open and hence \( f \) is a quasi \( \psi ag \)-open function.

Theorem: 3.6 A function \( f: X \to Y \) is quasi \( \psi ag \)-open if and only if \( f^{-1}(\text{Cl}(B)) \subset \psi ag - Cl(f^{-1}(B)) \) for every subset \( B \) of \( Y \).

**Proof:** Suppose that \( f \) is quasi \( \psi ag \)-open. For any subset \( B \) of \( Y \), \( f^{-1}(B) \subset \psi ag - Cl(f^{-1}(B)) \). Therefore by theorem 3.5, there exists a closed set \( F \) in \( Y \) such that \( B \subset F \) and \( f^{-1}(F) \subset Y \setminus F \). Hence, we obtain \( f(U) \subset Y \setminus F \). On the other hand, it follows that \( B \subset F \), \( Y \setminus F \subset Y \setminus B = f(U) \). Thus, we obtain \( f(U) \subset Y \setminus F \) which is open and hence \( f \) is quasi \( \psi ag \)-open function.

**Lemma:** 3.7 Let \( f: X \to Y \) and \( g: Y \to Z \) be two functions and \( g \circ f: X \to Z \) is quasi \( \psi ag \)-open. If \( g \) is continuous injective, then \( f \) is quasi \( \psi ag \)-open.

**Proof:** Let \( U \) be a \( \psi ag \)-open set in \( X \). Then \( (g \circ f)(U) \) is open in \( Z \) since \( g \circ f \) is quasi \( \psi ag \)-open. Again \( g \) is an injective continuous function, \( f(U) = f^{-1}(g \circ f)(U) \) is open in \( Y \). This shows that \( f \) is quasi \( \psi ag \)-open.

4. Quasi \( \psi ag \)-closed functions

**Definition:** 4.1 A function \( f: X \to Y \) is said to be quasi \( \psi ag \)-closed if the image of each \( \psi ag \)-closed set in \( X \) is closed in \( Y \).

Clearly, every quasi \( \psi ag \)-closed function is closed as well as \( \psi ag \)-closed.

**Remark:** 4.2 Every \( \psi ag \)-closed (resp. closed) function need not be quasi \( \psi ag \)-closed as shown by the following example.

**Example:** 4.3 Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a, b\}\} \). Define a function \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = b \) and \( f(b) = c \).
and \( f(c) = a \). Then clearly \( f \) is \( \psi ag \)-closed as well as closed but not quasi \( \psi ag \)-closed.

**Theorem: 4.4** Let \( X \) and \( Y \) be topological spaces. Then the function \( g: X \to Y \) is a quasi \( \psi ag \)-closed if and only if \( g(X) \) is closed in \( Y \) and \( g(V) \setminus g(X \setminus V) \) is open in \( g(X) \) whenever \( V \) is \( \psi ag \)-open in \( X \).

**Proof:** Necessity: Suppose \( g: X \to Y \) is a quasi \( \psi ag \)-closed function. Since \( X \) is \( \psi ag \)-closed, \( g(X) \) is closed in \( Y \) and \( g(V) \setminus g(X \setminus V) = g(V) \cap g(X) \) is open in \( g(X) \) when \( V \) is \( \psi ag \)-open in \( X \).

Sufficiency: Suppose \( g(X) \) is closed in \( Y \), \( g(V) \setminus g(X \setminus V) \) is open in \( g(X) \) when \( V \) is \( \psi ag \)-open in \( X \), and let \( C \) be closed in \( X \). Then \( g(C) = g(X) \setminus (g(X \setminus C) \setminus g(C)) \) is closed in \( X \) and hence, closed in \( Y \).

**Corollary: 4.5** Let \( X \) and \( Y \) be topological spaces and let \( g: X \to Y \) be a \( \psi ag \)-continuous quasi \( \psi ag \)-closed surjective function. Then the topology on \( Y \) is \( \{g(V) \setminus g(X \setminus V): V \text{ is } \psi ag \text{-open in } X \} \).

**Proof:** Let \( W \) be open in \( Y \). Then \( g^{-1}(W) \) is \( \psi ag \)-open in \( X \), and \( g(g^{-1}(W) \setminus g(X \setminus g^{-1}(W))) = W \). Hence, all open sets in \( Y \) are of the form \( g(V) \setminus g(X \setminus V) \). On the other hand, all sets of the form \( g(V) \setminus g(X \setminus V) \), \( V \) is \( \psi ag \)-open in \( X \), are open in \( Y \).

5. Separation axioms

In this section we introduce and study weak separation axioms such as \( \psi ag-T_0 \), \( \psi ag-T_1 \), \( \psi ag-T_2 \) spaces and obtain some of their properties.

**Definition: 5.1** A topological space \( X \) is said to be \( \psi ag-T_0 \) space if for each pair of distinct points \( x \) and \( y \) of \( X \), there exists a \( \psi ag \)-open set containing one point but not the other.

**Theorem: 5.2** A topological space \( X \) is a \( \psi ag-T_0 \) space if and only if \( \psi ag \)-closures of distinct points are distinct.

**Proof:** Let \( x \) and \( y \) be distinct points of \( X \). Since \( X \) is \( \psi ag-T_0 \) space, there exists a \( \psi ag \)-open set \( G \) such that \( x \in G \) and \( y \notin G \). Consequently, \( X \setminus G \) is a \( \psi ag \)-closed set containing \( y \) but not \( x \). But \( \psi ag Cl(A) \{y\} \) is the intersection of all \( \psi ag \)-closed sets containing \( y \). Hence \( y \notin \psi ag Cl(A) \{y\} \). But \( x \notin \psi ag Cl(A) \{y\} \) as \( x \notin X \setminus G \). Therefore, \( \psi ag Cl(A) \{x\} \neq \psi ag Cl(A) \{y\} \).

Conversely, let \( \psi ag Cl(A) \{x\} \neq \psi ag Cl(A) \{y\} \) for \( x \neq y \). Then there exists at least one point \( z \in X \) such that \( z \in \psi ag Cl(A) \{x\} \) but \( z \notin \psi ag Cl(A) \{y\} \). We claim \( x \notin \psi ag Cl(A) \{y\} \), because if \( x \in \psi ag Cl(A) \{y\} \) then \( \{y\} \subset \psi ag Cl(A) \{y\} \) implies \( \psi ag Cl(A) \{x\} \subset \psi ag Cl(A) \{y\} \). So \( z \notin \psi ag Cl(A) \{y\} \), which is a contradiction. Hence \( x \notin \psi ag Cl(A) \{y\} \), which implies \( x \in X \setminus \psi ag Cl(A) \{y\} \), which is a \( \psi ag \)-open set containing \( x \) but not \( y \). Hence \( X \) is \( \psi ag-T_0 \) space.

**Theorem: 5.3** If \( f: X \to Y \) is a bijection strongly \( \psi ag \)-open and \( X \) is \( \psi ag-T_0 \) space, then \( Y \) is also \( \psi ag-T_0 \) space.

**Proof:** Let \( y_1 \) and \( y_2 \) be two distinct points of \( Y \). Since \( f \) is bijective there exist distinct points \( x_1 \) and \( x_2 \) of \( X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Since \( X \) is \( \psi ag-T_0 \) space there exists a \( \psi ag \)-open set \( G \) such that \( x_1 \in G \) and \( x_2 \notin G \). Therefore \( y_1 = f(x_1) \in f(G) \) and \( y_2 = f(x_2) \notin f(G) \). Since \( f \) being strongly \( \psi ag \)-open function, \( f(G) \) is \( \psi ag \)-open in \( Y \). Thus, there exists a \( \psi ag \)-open set \( f(G) \) in \( Y \) such that \( y_1 \notin f(G) \) and \( y_2 \in f(G) \). Therefore \( Y \) is \( \psi ag-T_0 \) space.

**Definition: 5.4** A topological space \( X \) is said to be \( \psi ag-T_1 \) space if for any pair of distinct points \( x \) and \( y \), there exists a \( \psi ag \)-open sets \( G \) and \( H \) such that \( x \in G \), \( y \notin G \) and \( x \notin H \), \( y \in H \).

**Theorem: 5.5** A topological space \( X \) is \( \psi ag-T_1 \) space if and only if singletons are \( \psi ag \)-closed sets.

**Proof:** Let \( X \) be a \( \psi ag-T_1 \) space and \( x \in X \). Let \( y \in X \setminus \{x\} \). Then for \( x \neq y \), there exists \( \psi ag \)-open set \( U_y \) such that \( y \in U_y \) and \( x \notin U_y \). Consequently, \( y \in U_y \subset X \setminus \{x\} \). That is \( X \setminus \{x\} = \cup \{U_y : y \in X \setminus \{x\}\} \), which is \( \psi ag \)-open set. Hence \( \{x\} \) is \( \psi ag \)-closed set.

Conversely, suppose \( \{x\} \) is \( \psi ag \) closed set for every \( x \in X \). Let \( x \) and \( y \) \( X \) with \( x \neq y \). Now \( x \neq y \) implies \( y \in X \setminus \{x\} \). Hence \( X \setminus \{x\} \) is \( \psi ag \)-open set containing \( y \) but not \( x \). Similarly, \( X \setminus \{y\} \) is \( \psi ag \)-open set containing \( x \) but not \( y \). Therefore \( X \) is \( \psi ag-T_1 \) space.

**Theorem: 5.6** The property being \( \psi ag-T_1 \) space is preserved under bijection and strongly \( \psi ag \)-open function.

**Proof:** Let \( f: X \to Y \) be bijection and strongly \( \psi ag \)-open function. Let \( X \) be a \( \psi ag-T_1 \) space and \( y_1, y_2 \) be any two distinct points of \( Y \). Since \( f \) is bijective there exist distinct points \( x_1, x_2 \) of \( X \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Now \( X \) being a \( \psi ag-T_1 \) space, there exist \( \psi ag \)-open sets \( G \) and \( H \) such that \( x_1 \in G \), \( x_2 \in G \) and \( x_1 \notin H \), \( x_2 \in H \). Therefore \( y_1 = f(x_1) \in f(G) \) but \( y_2 = f(x_2) \notin f(G) \) and \( y_2 = f(x_2) \in f(H) \) and \( y_1 = f(x_1) \notin f(H) \). Now \( f \) being strongly...
\( \psi ag \)-open, \( f(G) \) and \( f(H) \) are \( \psi ag \)-open subsets of \( Y \) such that \( y_1 \in f(G) \) but \( y_2 \notin f(G) \) and \( y_2 \in f(H) \) and \( y_1 \notin f(H) \). Hence \( Y \) is \( \psi ag-T_1 \)-space.

**Theorem:** 5.7 Let \( f:X \to Y \) be bijective and \( \psi ag \)-open function. If \( X \) is \( \psi ag-T_1 \) and \( T_{\psi ag} \)-space, then \( Y \) is \( \psi ag-T_1 \)-space.

**Proof:** Let \( y_1, y_2 \) be any two distinct points of \( Y \). Since \( f \) is bijective there exist distinct points \( x_1, x_2 \) of \( X \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Now \( X \) being a \( \psi ag-T_1 \)-space, there exist \( \psi ag \)-open sets \( G \) and \( H \) such that \( x_1 \in G \) and \( x_2 \in H \). Therefore \( y_1 \in f(G) \) but \( y_2 \notin f(G) \) and \( y_2 \in f(H) \) and \( y_1 \notin f(H) \). Now \( X \) is \( T_{\psi ag} \)-space which implies \( G \) and \( H \) are open sets in \( X \) and \( f \) is \( \psi ag \)-open function, \( f(G) \) and \( f(H) \) are \( \psi ag \)-open subsets of \( Y \). Thus there exist \( \psi ag \)-open sets such that \( y_1 \notin f(G) \) but \( y_2 \notin f(G) \) and \( y_2 \in f(H) \) and \( y_1 \notin f(H) \). Hence \( Y \) is \( \psi ag-T_1 \)-space.

**Theorem:** 5.8 If \( f:X \to Y \) is \( \psi ag \)-continuous injection and \( Y \) is \( T_1 \) then \( X \) is \( \psi ag-T_1 \)-space.

**Proof:** Let \( f:X \to Y \) be \( \psi ag \)-continuous injection and \( Y \) is \( T_1 \). For any two distinct points \( x_1, x_2 \) of \( X \) there exist distinct points \( y_1, y_2 \) of \( Y \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Since \( Y \) is \( T_1 \)-space, there exists open sets \( U \) and \( V \) in \( Y \) such that \( y_1 \notin U \), \( y_2 \notin U \) and \( y_1 \in V \), \( y_2 \in V \). That is, \( x_1 \in f^{-1}(U) \), \( x_2 \in f^{-1}(V) \), \( x_2 \notin f^{-1}(V) \). Since \( f \) is \( \psi ag \)-continuous \( f^{-1}(U) \), \( f^{-1}(V) \) are \( \psi ag \)-open sets in \( X \). Thus, for two distinct points \( x_1, x_2 \) of \( X \) there exist \( \psi ag \)-open sets \( f^{-1}(U) \) and \( f^{-1}(V) \) such that \( x_1 \in f^{-1}(U) \), \( x_2 \in f^{-1}(V) \), \( x_2 \notin f^{-1}(V) \). Therefore \( X \) is \( \psi ag-T_1 \)-space.

**Theorem:** 5.9 If \( f:X \to Y \) is \( \psi ag \)-irresolute injection function and \( Y \) is \( \psi ag-T_1 \)-space then \( X \) is \( \psi ag-T_3 \)-space.

**Proof:** Let \( x_1, x_2 \) be a pair of distinct points in \( X \). Since \( f \) is injective there exist distinct points \( y_1, y_2 \) of \( Y \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Since \( Y \) is \( \psi ag-T_3 \)-space, there exists \( \psi ag \)-open sets \( U \) and \( V \) in \( Y \) such that \( y_1 \notin U \) and \( y_2 \notin V \). Clearly, \( X \) is \( \psi ag-T_3 \)-space.

\[ \psi ag \text{-open sets} \]

\[ \psi ag \text{-irresolute} \]

\[ \psi ag \text{-closed} \]

\[ \psi ag \text{-open set} \]

\[ \psi ag \text{-closed set} \]

\[ \psi ag \text{-irresolute} \]

\[ \psi ag \text{-continuous} \]

\[ \psi ag \text{-bijective} \]

**Definition:** 5.10 A topological space \( X \) is said to be \( \psi ag-T_2 \)-space if for any pair of distinct points \( x \) and \( y \), there exists disjoint \( \psi ag \)-open sets \( G \) and \( H \) such that \( x \in G \) and \( y \in H \).

**Theorem:** 5.11 If \( f:X \to Y \) is \( \psi ag \)-continuous injection and \( Y \) is \( T_2 \) then \( X \) is \( \psi ag-T_2 \)-space.

**Proof:** Let \( f:X \to Y \) be \( \psi ag \)-continuous injection and \( Y \) is \( T_2 \). For any two distinct points \( x_1, x_2 \) of \( X \) there exist distinct points \( y_1, y_2 \) of \( Y \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Since \( Y \) is \( T_2 \)-space, there exists distinct open sets \( U \) and \( V \) in \( Y \) such that \( y_1 \notin U \) and \( y_2 \notin V \). That is, \( x_1 \in f^{-1}(U) \), \( x_2 \in f^{-1}(V) \). Since \( f \) is \( \psi ag \)-continuous \( f^{-1}(U) \), \( f^{-1}(V) \) are \( \psi ag \)-open sets in \( X \). Further \( f \) is injective, \( f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset \). Thus, for two distinct points \( x_1, x_2 \) of \( X \) there exist disjoint \( \psi ag \)-open sets \( f^{-1}(U) \) and \( f^{-1}(V) \) such that \( x_1 \notin f^{-1}(U) \) and \( x_2 \notin f^{-1}(V) \). Therefore \( X \) is \( \psi ag-T_2 \)-space.

**Theorem:** 5.12 If \( f:X \to Y \) is \( \psi ag \)-irresolute injection function and \( Y \) is and \( \psi ag-T_2 \)-space then \( X \) is \( \psi ag-T_2 \)-space.

**Proof:** Let \( x_1, x_2 \) be a pair of distinct points in \( X \). Since \( f \) is injective there exist distinct points \( y_1, y_2 \) of \( Y \) such that \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Since \( Y \) is \( \psi ag-T_2 \)-space, there exists \( \psi ag \)-open sets \( U \) and \( V \) in \( Y \) such that \( y_1 \notin U \) and \( y_2 \notin V \). Clearly, \( X \) is \( \psi ag-T_2 \)-space.

**Theorem:** 5.13 In any topological space the following are equivalent:

1. \( X \) is \( \psi ag-T_2 \)-space,
2. For each \( x \neq y \), there exists a \( \psi ag \)-open set \( U \) such that \( x \in U \) and \( y \notin \psi agCl(U) \),
3. For each \( x \in X \), \( \{x\} = \psi agCl(U) \) if \( U \) is a \( \psi ag \)-open set in \( X \) and \( x \in U \).

**Proof:** (1) \( \Rightarrow \) (2): Assume (1) holds. Let \( x \in X \) and \( x \neq y \), then there exist disjoint \( \psi ag \)-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \notin V \). Clearly, \( X \) is \( \psi ag \)-closed set. Since \( U \cap V = \emptyset \), \( U \subseteq X \setminus V \). Therefore \( \psi agCl(U) \subseteq \psi agCl(X \setminus V) = X \setminus V \). Now \( y \notin X \setminus V \) implies \( y \notin \psi agCl(U) \).

**Proof:** (2) \( \Rightarrow \) (3): For each \( x \neq y \), there exists a \( \psi ag \)-open set \( U \) such that \( x \in U \) and \( y \notin \psi agCl(U) \). So \( y \notin \psi agCl(U) \) if \( U \) is a \( \psi ag \)-open set in \( X \) and \( x \in U \) = \( \{x\} \).
(3) ⇒ (1): Let $x, y \in X$ and $x \neq y$. By hypothesis there exists a $\psi ag$-open set $U$ such that $x \in U$ and $\psi ag Cl(U)$. This implies there exists a $\psi ag$-closed set $V$ such that $y \notin V$. Therefore $y \in X - V$ and $X - V$ is $\psi ag$-open set. Thus, there exist two disjoint $\psi ag$-open sets $U$ and $X - V$ such that $x \in U$ and $y \in X - V$. Therefore $X$ is $\psi ag - T_2$-space.

References